

BOUNDED AND ALMOST PERIODIC SOLUTIONS FOR SECOND ORDER DIFFERENTIAL EQUATION INVOLVING REFLECTION OF THE ARGUMENT

DAXIONG PIAO [†] [‡] AND NA XIN [‡]

ABSTRACT. In this paper we investigate the existence and uniqueness of bounded, periodic and almost periodic solutions for second order differential equations involving reflection of the argument. The relationship between frequency modules of forced term and the solution of the equation is considered.

1. INTRODUCTION

The differential equations involving reflection of argument have applications in the study of stability of differential-difference equations, see Sarkovskii [1], and such equations show very interesting properties by themselves, so many authors worked on them. Wiener and Aftabizadeh [2] initiated to study boundary value problems for the second order differential equations involving reflection of the argument. Gupta [3,4] investigated two point boundary value problems for this kind of equations under the Caratheodory conditions. Afidabizadeh, Huang, and Wiener [5] studied the existence of unique bounded solution of first order equation

$$\dot{x}(t) = f(t, x(t), x(-t)).$$

They proved that $x(t)$ is almost periodic by assuming the existence of bounded solution $x(t)$ of the equation. In [6,7], One of the present authors investigated existence and uniqueness of periodic, almost periodic and pseudo almost periodic solutions of the equations

$$\dot{x}(t) + ax(t) + bx(-t) = g(t), b \neq 0, t \in R,$$

and

$$\dot{x}(t) + ax(t) + bx(-t) = f(t, x(t), x(-t)), b \neq 0, t \in R.$$

Key words and phrases. Bounded solution, Almost Periodic Solution, Second Order Differential Equation, Reflection of Argument.

2000 *Mathematics Subject Classification* 34C27, 34K13, 34K14.

[‡] Corresponding author. Supported by the NSF of Shangdong Province (No.ZR2012AM018).

In [12], the authors studied the first order operator $x'(t) + mx(-t)$ coupled with periodic boundary value conditions, and described the eigenvalues of the operator and obtained the expression of its related Greens function in the non resonant case.

Our present paper is motivated by above references, and devoted to investigate the existence and uniqueness of bounded, periodic and almost periodic solution of the second order linear equation

$$(1.1) \quad \ddot{x}(t) + ax(t) + bx(-t) = g(t), b \neq 0, t \in \mathbf{R},$$

and nonlinear equation

$$(1.2) \quad \ddot{x}(t) + ax(t) + bx(-t) = f(t, x(t), x(-t)), b \neq 0, t \in \mathbf{R},$$

respectively, where g and f satisfy some assumptions which will be stated later.

Our paper is organized as following. In Section 2, we state some lemmas and basic formulations; In Section 3, we study the case $a < b < -a$ for the equations (1.1) and (1.2); In Section 4, we study the case $-a < b < a$.

2. SOME LEMMAS AND USEFUL FORMULATIONS

Now we give some definitions for our business.

Definition 2.1. ^[8-9] A function $f : \mathbf{R} \rightarrow \mathbf{C}$ is almost periodic, if the ϵ -translation of f

$$\mathbf{T}(f, \epsilon) = \{\tau \in \mathbf{R} : |f(t + \tau) - f(t)| < \epsilon, \forall t \in \mathbf{R}\}$$

is relatively dense in \mathbf{R} . We denote the set all such functions by $AP(\mathbf{R})$.

Definition 2.2. ^[8-9] A function $F : \mathbf{R} \times \mathbf{C}^2 \rightarrow \mathbf{R}$ is almost periodic for t uniformly on \mathbf{C}^2 , if for any compact $\mathbf{W} \subset \mathbf{C}^2$, the ϵ -translation of F

$$\mathbf{T}(F, \epsilon, \mathbf{W}) = \{\tau \in \mathbf{R} : |F(t + \tau, x, y) - F(t, x, y)| < \epsilon, \forall (t, x, y) \in \mathbf{R} \times \mathbf{W}\}$$

is relatively dense in \mathbf{R} . We denote the set of all such functions by $AP(\mathbf{R} \times \mathbf{C}^2)$. \mathbf{R} is the set of all real numbers, and \mathbf{C} is the set of all complex numbers.

we state two useful lemmas those can be easily proven.

Lemma 2.3. *If $g(t) \in AP(\mathbf{R})$, then $g(-t) \in AP(\mathbf{R})$. Furthermore if τ is an ϵ -translation of $g(t)$, then τ is also an ϵ -translation of $g(-t)$. If $g(t)$ is ω -periodic, then $g(-t)$ is also ω -periodic.*

Lemma 2.4. *If $g(t) \in AP(\mathbf{R})$, then $\text{mod}(g(t)) = \text{mod}(g(-t))$ and $\text{Freq}(g(t)) = -\text{Freq}(g(-t))$, where $\text{Freq}(g)$ denotes the frequency set of $g(t)$.*

We refer the readers to good books [8-10] for the basic results on the almost periodic functions .

Before treating the nonlinear equation (1.2), we need consider the linear equation (1.1) first .

Let $x_1 = x(t)$, $x_2 = x_1(-t)$, $x_3 = \dot{x}_1(t)$, $x_4 = \dot{x}_3(-t)$, then we obtain a system

$$(2.1) \quad \begin{cases} \dot{x}_1 = x_3 \\ \dot{x}_2 = -x_4 \\ \dot{x}_3 = -ax_1 - bx_2 + g(t) \\ \dot{x}_4 = bx_1 + ax_2 - g(-t) \end{cases}$$

or

$$(2.2) \quad \dot{\mathbf{x}}(t) = A\mathbf{x} + \mathbf{g}(t)$$

$$\text{where } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -a & -b & 0 & 0 \\ b & a & 0 & 0 \end{pmatrix} \text{ and } \mathbf{g}(t) = \begin{pmatrix} 0 \\ 0 \\ g(t) \\ -g(-t) \end{pmatrix}.$$

Now $\det(rI - A) = r^4 + 2ar^2 + a^2 - b^2$, so the eigenvalues of A are $r_1 = \alpha = \sqrt{b-a}$, $r_2 = -\alpha = -\sqrt{b-a}$, $r_3 = \beta = \sqrt{-a-b}$, $r_4 = -\beta = -\sqrt{-a-b}$. If $\alpha\beta \neq 0$, then their corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ \alpha \\ \alpha \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \\ \alpha \\ \alpha \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} -1 \\ -1 \\ -\beta \\ \beta \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} 1 \\ 1 \\ -\beta \\ \beta \end{pmatrix} \text{ re-}$$

spectively. So the linear transformation $x = Py$, where $P = \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ \alpha & \alpha & -\beta & -\beta \\ \alpha & \alpha & \beta & \beta \end{pmatrix}$,

turn the equation (1.3) into

$$(2.3) \quad \dot{\mathbf{y}}(t) = B\mathbf{y} + \mathbf{f}(t),$$

where $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$, $B = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & -\beta \end{pmatrix}$ and $\mathbf{f}(t) = P\mathbf{g}(t) =$

$$\begin{pmatrix} -g(t) - g(-t) \\ -g(t) - g(-t) \\ \beta(-g(t) + g(-t)) \\ \beta(g(t) - g(-t)) \end{pmatrix}.$$

3. CASE 1: $a < b < -a$

By the standard formulation we can obtain following lemma.

Lemma 3.1. *Suppose that $g(t) \in C(\mathbf{R})$, and bounded on \mathbf{R} , and $\alpha > 0, \beta > 0$. Then the general solution of the system (1.4) on \mathbf{R} , is given by*

$$(3.1) \quad \mathbf{y} = e^{Bt}\mathbf{c} + Y(t),$$

where $\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}$, and $Y(t) = \begin{pmatrix} \int_t^\infty e^{\alpha(t-s)}(g(s) + g(-s))ds \\ -\int_{-\infty}^t e^{-\alpha(t-s)}(g(s) + g(-s))ds \\ -\beta \int_t^\infty e^{\beta(t-s)}(-g(s) + g(-s))ds \\ \beta \int_{-\infty}^t e^{-\beta(t-s)}(g(s) - g(-s))ds \end{pmatrix}$ and

so the general solution of the system (1.2) or (1.3) on \mathbf{R} , is given by

$$(3.2) \quad \mathbf{x} = P\mathbf{y} = P(e^{Bt}\mathbf{c} + Y(t))$$

or

$$(3.3) \quad x_1 = c_1 e^{\alpha t} - c_2 e^{-\alpha t} - c_3 e^{\beta t} + c_4 e^{-\beta t} + w_1(t)$$

$$(3.4) \quad x_2 = -c_1 e^{\alpha t} + c_2 e^{-\alpha t} - c_3 e^{\beta t} + c_4 e^{-\beta t} + w_2(t)$$

$$(3.5) \quad x_3 = -c_1 \alpha e^{\alpha t} + c_2 \alpha e^{-\alpha t} - c_3 \beta e^{\beta t} + c_4 \beta e^{-\beta t} + w_3(t)$$

$$(3.6) \quad x_4 = -c_1 \alpha e^{\alpha t} + c_2 \alpha e^{-\alpha t} + c_3 \beta e^{\beta t} + c_4 \beta e^{-\beta t} + w_4(t)$$

where

$$\begin{aligned}
w_1(t) &= \int_t^\infty e^{\alpha(t-s)}(g(s) + g(-s))ds \\
&+ \int_{-\infty}^t e^{-\alpha(t-s)}(g(s) + g(-s))ds \\
&+ \beta \int_t^\infty e^{\beta(t-s)}(-g(s) + g(-s))ds \\
&+ \beta \int_{-\infty}^t e^{-\beta(t-s)}(g(s) - g(-s))ds.
\end{aligned}$$

$$\begin{aligned}
w_2(t) &= \int_t^\infty e^{\alpha(t-s)}(-g(s) - g(-s))ds \\
&+ \int_{-\infty}^t e^{-\alpha(t-s)}(-g(s) - g(-s))ds \\
&+ \beta \int_t^\infty e^{\beta(t-s)}(-g(s) + g(-s))ds \\
&+ \beta \int_{-\infty}^t e^{-\beta(t-s)}(g(s) - g(-s))ds.
\end{aligned}$$

$$\begin{aligned}
w_3(t) &= \alpha \int_t^\infty e^{\alpha(t-s)}(g(s) - g(-s))ds \\
&- \alpha \int_{-\infty}^t e^{-\alpha(t-s)}(-g(s) - g(-s))ds \\
&+ \beta^2 \int_t^\infty e^{\beta(t-s)}(-g(s) + g(-s))ds \\
&- \beta^2 \int_{-\infty}^t e^{-\beta(t-s)}(g(s) - g(-s))ds.
\end{aligned}$$

$$\begin{aligned}
w_4(t) &= \alpha \int_t^\infty e^{\alpha(t-s)}(g(s) + g(-s))ds \\
&- \int_{-\infty}^t e^{-\alpha(t-s)}(g(s) + g(-s))ds \\
&- \beta^2 \int_t^\infty e^{\beta(t-s)}(-g(s) + g(-s))ds \\
&+ \beta^2 \int_{-\infty}^t e^{-\beta(t-s)}(g(s) - g(-s))ds.
\end{aligned}$$

Because there are four arbitrary constants in (3.3), we can not conclude that (3.2) is the general solution of (1.1). (3.3) may not even be a solution of (1.1) for some constants c_1, c_2, c_3 and c_4 indeed.

Lemma 3.2. *Let $g(t) \in C(\mathbf{R})$ is bounded on \mathbf{R} , and $\alpha > 0, \beta > 0$, then every solution of Eq.(1.1) is of the form (3.3), if and only if $c_1 = c_2, c_3 = -c_4$, that is the general solution of Eq.(1.1) is of the form*

$$(3.7) \quad x(t) = k_1(e^{\alpha t} - e^{-\alpha t}) + k_2(e^{\beta t} + e^{-\beta t}) + w_1(t)$$

where k_1, k_2 are arbitrary constants.

Proof From the requirements of $x_1 = x(t)$, $x_2 = x_1(-t)$, $x_3 = \dot{x}_1(t)$, $x_4 = x_3(-t)$, we can derive $c_1 = c_2, c_3 = -c_4$. Let $k_1 = c_1 = c_2, k_2 = -c_3 = c_4$.

Theorem 3.3. *Under the condition of Lemma (3.1), Eq.(1.1) has a unique bounded solution given by*

$$x(t) = w_1(t) = \int_t^\infty e^{\alpha(t-s)}(g(s) + g(-s))ds + \int_{-\infty}^t e^{-\alpha(t-s)}(g(s) + g(-s))ds \\ + \beta \int_t^\infty e^{\beta(t-s)}(-g(s) + g(-s))ds + \beta \int_{-\infty}^t e^{-\beta(t-s)}(g(s) - g(-s))ds.$$

and moreover, $\sup_{t \in \mathbf{R}} |x(t)| \leq (\frac{2}{\alpha} + 1) \sup_{t \in \mathbf{R}} |g(t)|$

Proof (3.11) is a bounded solution if and only if $k_1 = k_2 = 0$. We obtain the inequality $\sup_{t \in \mathbf{R}} |x(t)| \leq (\frac{2}{\alpha} + 1) \sup_{t \in \mathbf{R}} |g(t)|$ by evaluating $x(t) = w_1(t)$.

Theorem 3.4. *Let $g(t) \in AP(\mathbf{R})$, then Eq.(1.1) has a unique almost periodic solution $x(t)$, and $\text{mod}(x) = \text{mod}(g)$. Furthermore, if $g(t)$ is periodic, then Eq.(1.1) has a unique harmonic solution.*

Proof We will show $x(t) = w_1(t)$ is almost periodic solution of (1.1). For $\tau \in T(g(t), \epsilon)$,

$$\begin{aligned}
& |x(t+\tau) - x(t)| \\
&= \left| \int_{t+\tau}^{\infty} e^{\alpha(t+\tau-s)}(g(s) + g(-s))ds - \int_{-\infty}^{t+\tau} e^{-\alpha(t+\tau-s)}(-g(s) - g(-s))ds \right. \\
&\quad + \beta \int_{t+\tau}^{\infty} e^{\beta(t+\tau-s)}(-g(s) + g(-s))ds + \beta \int_{-\infty}^{t+\tau} e^{-\beta(t+\tau-s)}(g(s) - g(-s))ds \\
&\quad - \int_t^{\infty} e^{\alpha(t-s)}(g(s) + g(-s))ds + \int_{-\infty}^t e^{-\alpha(t-s)}(-g(s) - g(-s))ds \\
&\quad \left. - \beta \int_t^{\infty} e^{\beta(t-s)}(-g(s) + g(-s))ds - \beta \int_{-\infty}^t e^{-\beta(t-s)}(g(s) - g(-s))ds \right| \\
&\leq \int_t^{\infty} e^{\alpha(t-s)}(|g(s+\tau) - g(s)| + |g(-s-\tau) - g(-s)|)ds \\
&\quad + \int_{-\infty}^t e^{-\alpha(t-s)}(|g(s+\tau) - g(s)| + |g(-s-\tau) - g(-s)|)ds \\
&\quad + \beta \int_t^{\infty} e^{\beta(t-s)}(|g(s+\tau) - g(s)| + |g(-s-\tau) - g(-s)|)ds \\
&\quad + \beta \int_{-\infty}^t e^{-\beta(t-s)}(|g(s+\tau) - g(s)| + |g(-s-\tau) - g(-s)|)ds \\
&\leq 4\left(\frac{1}{\alpha} + 1\right)\epsilon.
\end{aligned}$$

So $x(t) \in AP(\mathbf{R})$, and $\text{mod}(x) \subset \text{mod}(g)$. From $g(t) = \ddot{x}(t) + ax(t) + bx(-t)$, and the lemma 2.4 we conclude $\text{mod}(g) \subset \text{mod}(x)$, and so $\text{mod}(x) = \text{mod}(g)$. If $g(t)$ is ω -periodic, then $|x(t+\omega) - x(t)| \leq 4\left(\frac{1}{\alpha} + 1\right)\|g(t+\omega) - g(t)\| = 0$. So $x(t)$ is ω -periodic solution.

Uniqueness. If there is another almost periodic solution $x_1(t)$ for Eq.(3), then the difference $x(t) - x_1(t)$ should be a solution of the homogeneous equation

$$(3.8) \quad \ddot{x}(t) + ax(t) + bx(-t) = 0, b \neq 0, t \in \mathbf{R}.$$

According to the Lemma 3.12, we can derive

$$(3.9) \quad x(t) - x_1(t) = k_1(e^{\alpha t} - e^{-\alpha t}) + k_2(e^{\beta t} - e^{-\beta t})$$

for some constant k_1, k_2 . If $|k_1| + |k_2| \neq 0$, then $x(t) - x_1(t)$ will be unbounded. This is a contradiction to the boundedness of almost periodic function. So $x(t) - x_1(t) \equiv 0$, i.e $x(t) \equiv x_1(t)$.

Theorem 3.5. Suppose $f(t, x, y)$ is almost periodic on t uniformly with respect to x and y on any compact set $\mathbf{W} \subset \mathbf{C}^2$, and satisfies Lipschitz

condition

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq L(|x_1 - x_2| + |y_1 - y_2|)$$

for any $(x_1, y_1), (x_2, y_2) \in \mathbf{W}$, where $L < \frac{\alpha}{4(1+\alpha)}$. Then Eq.(1.2) has a unique almost periodic solution $x(t)$ and $\text{mod}(x) = \text{mod}(f)$. In addition, if f is periodic in t , then Eq.(1.2) has a unique harmonic solution $x(t)$.

Proof We know the subset

$$\mathbf{B} = \{\phi(t) : \phi \in AP(\mathbf{R}), \text{mod}(\phi) \subset \text{mod}(f)\}$$

of $AP(\mathbf{R})$ is a Banach space with the supremum norm $\|\phi\| = \sup_{t \in \mathbf{R}} |\phi(t)|$. For any $\phi \in \mathbf{B}$, we know $f(t, \phi(t), \phi(-t)) \in \mathbf{B}$ [11]. According to the theorem 2.1 we see the equation

$$(3.10) \quad \ddot{x}(t) + ax(t) + bx(-t) = f(t, \phi(t), \phi(-t)), b \neq 0, t \in \mathbf{R}$$

possess a unique almost periodic solution, denote it by $(T\phi)(t)$. Then we define a mapping $T : \mathbf{B} \rightarrow \mathbf{B}$. Now we show T is contracted.

For $\phi(t), \psi(t) \in \mathbf{B}$, the equation

$$(3.11) \quad \ddot{x}(t) + ax(t) + bx(-t) = f(t, \phi(t), \phi(-t)) - f(t, \psi(t), \psi(-t)), b \neq 0, t \in \mathbf{R}$$

has a unique almost periodic solution $(T\phi - T\psi)(t)$, and

$$\begin{aligned} & |(T\phi - T\psi)(t)| \\ &= \left| \int_t^\infty e^{\alpha(t-s)} [f(s, \phi(s), \phi(-s)) - f(s, \psi(s), \psi(-s))] \right. \\ &\quad \left. + f(-s, \phi(-s), \phi(s)) - f(-s, \psi(-s), \psi(s))] ds \right. \\ &+ \left| \int_{-\infty}^t e^{-\alpha(t-s)} [f(s, \phi(s), \phi(-s)) - f(s, \psi(s), \psi(-s))] \right. \\ &\quad \left. + f(-s, \phi(-s), \phi(s)) - f(-s, \psi(-s), \psi(s))] ds \right. \\ &+ \left| \beta \int_t^\infty e^{\beta(t-s)} [-f(s, \phi(s), \phi(-s)) - f(s, \psi(s), \psi(-s))] \right. \\ &\quad \left. + f(-s, \phi(-s), \phi(s)) - f(-s, \psi(-s), \psi(s))] ds \right. \\ &+ \left| \beta \int_{-\infty}^t e^{-\beta(t-s)} [f(s, \phi(s), \phi(-s)) - f(s, \psi(s), \psi(-s))] \right. \\ &\quad \left. - f(-s, \phi(-s), \phi(s)) - f(-s, \psi(-s), \psi(s))] ds \right| \\ &\leq 4L\left(\frac{1}{\alpha} + 1\right) \|\phi - \psi\|. \end{aligned}$$

So

$$\|T\phi - T\psi\| \leq 4L\left(\frac{1}{\alpha} + 1\right) \|\phi - \psi\|.$$

Since $L < \frac{\alpha}{4(1+\alpha)}$, T is a contraction mapping, and so T has a unique fixed point in B . That is to say the equation (1.1) has a unique almost periodic solution $x(t)$ and $\text{mod}(x) \subset \text{mod}(f)$.

If $f(t, x, y)$ is continuous ω -periodic in t , then for any ω -periodic function $\phi(t)$, $f(t, \phi(t), \phi(-t))$ is continuous ω -periodic function too. We denote by C_ω the set of all continuous ω -periodic functions. Then we know C_ω is a Banach space with supremum norm $\|\phi\| = \sup_{t \in \mathbb{R}} |\phi(t)|$. From theorem 2.1, we conclude, for any $\phi(t) \in C_\omega$, Eq.(2.4) has a unique ω -periodic solution $T\phi \in C_\omega$. We can easily prove as above that T is contracted. So T has a unique fixed point $x(t) \in C_\omega$, i.e. there is a unique harmonic solution for Eq.(1.1). So the theorem 2.2 is completed.

4. CASE 2: $-a < b < a$

In this case, both α and β are pure imaginary numbers. Set $\alpha = i\mu$ and $\beta = i\nu$. By the standard formulation we can obtain

Lemma 4.1. *Suppose that $g(t) \in C(\mathbf{R})$, and $-a < b < a$. Then the general solution of the system (2.3) on \mathbf{R} , is given by*

$$(4.1) \quad \mathbf{y} = e^{Bt} \left(\mathbf{c} + \int_0^t e^{-Bs} \mathbf{f}(s) ds \right),$$

and so the general solution of (2.2) is given by

$$(4.2) \quad \mathbf{x} = P\mathbf{y} = Pe^{Bt} \left(\mathbf{c} + \int_0^t e^{-Bs} \mathbf{f}(s) ds \right)$$

where $B = \text{diag}\{i\mu, -i\mu, i\nu, -i\nu\}$.

Similar to section 3 we can derive that the general solution of Eq.(1.2) take the form

$$(4.3) \quad \begin{aligned} x(t) = & k_1(e^{i\mu t} - e^{-i\mu t}) + k_2(e^{i\nu t} + e^{-i\nu t}) - e^{i\mu t} \int_0^t e^{-i\mu s} (g(s) + g(-s)) ds \\ & + e^{-i\mu t} \int_0^t e^{i\mu s} (g(s) + g(-s)) ds - \beta e^{i\nu t} \int_0^t e^{-i\nu s} (-g(s) + g(-s)) ds \\ & + \beta e^{-i\nu t} \int_0^t e^{i\nu s} (g(s) - g(-s)) ds \end{aligned}$$

To this end, we introduce a result of Favard.

Lemma 4.2. ^[8] Assume the fourier series of almost function $f(t)$ is $\sum A_n e^{i\Lambda_n t}$. If there is $\alpha > 0$ such that $\Lambda_n > \alpha$, for any $n \in Z$, then the indefinite integral of $f(t)$ is almost periodic

Then we have following theorem.

Theorem 4.3. If $\forall \lambda_k \in \text{Freq}(g)$ s.t $|\mu \pm \lambda_k| \geq \rho > 0$, $|\nu \pm \lambda_k| \geq \sigma > 0$ and $-a < b < a$, where ρ and σ are some constants, then (1) every solution of (1.1) is almost periodic; (2) there exists a unique almost periodic solution $x(t)$ satisfies initial condition $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$

Proof (1) Lemma 2.4 and Lemma 4.2 yield those indefinite integrals in (4.3) are almost periodic.

(2) A solution $x(t)$ s.t $x(0) = x_0, \dot{x}(0) = \dot{x}_0$ if and only if $k_1 = \dot{x}_0/2i\mu$ and $k_2 = x_0/2$ in (4.3).

Remark 4.4. The existence of the term $bx(-t)$ with reflection of the argument in (1.1) may influence the boundeness of the solutions of the equation without reflection of the argument drastically.

Example 1 The general solution of second order equation

$$(4.4) \quad \ddot{x} + 4x = \cos 2t$$

is

$$x(t) = c_1 \cos 2t + c_2 \sin 2t + \frac{t}{4} \sin 2t.$$

So the equation (4.3) has no bounded solution. But according to Theorem 4.1, every solution of the equation

$$\ddot{x}(t) + 4x(t) + 2x(-t) = \cos 2t$$

is bounded, and almost periodic.

Example 2 The general solution of the equation

$$(4.5) \quad \dot{x}(t) + 2x(t) = \cos t$$

is $x(t) = c_1 \cos \sqrt{2}t + c_2 \sin \sqrt{2}t + \cos t$, so every solution of (4.5) is bounded. But from formulae (4.3), we can derive every solution of

$$\dot{x}(t) + 2x(t) + x(-t) = \cos t$$

is unbounded. We note that $\{-1, 1\} \subset \text{Freq}(g) = \text{Freq}(\cos t)$, and $\mu = \sqrt{a-b} = 1$. This shows the condition $\rho > 0, \sigma > 0$ of Theorem 4.2 is sharp.

Remark 4.5. One can investigate the solution of (1.1) and (1.2) for the other case of a and b . It seems that the study for the case $a = b$ or $a = -b$ is complicated. To study the solutions for nonlinear equation

$$x'' + x^3 + x(-t) = p(t)$$

may be vary interesting problem.

REFERENCES

1. A.N.Sharkovskii, Functional-differential equations with a finite group of argument transformations, in Asymptotic Behavior of Solutions of Functional-Differential Equations(Akad. Nauk Ukrain., Inst. Mat.,Kiev,(1978) pp.118-142.
2. A.R.Afidabizadeh, J.Wiener, Boundary value problems for differential equations with reflection of argument. Int. J. Math. Math.Sci.**8**(1985), 151–163.
3. C.P.Gupta, Existence and uniqueness theorem for boundary value problems involving reflection of the argument, Nonlinear Analysis, TAM, **11**(1987), 1075–1083.
4. C.P.Gupta, Two point boundary value problems involving reflection of the argument, Int. J. Math. Math. Sci. **10** (1987), 361–371.
5. A.R.Afidabizadeh, Y.K.Huang, and J.Wiener, Bounded solutions for Differential Equations with Reflection of the Argument, J. Math. Anal. Appl. **135** (1988),31–37.
6. D.Piao, Periodic and almost periodic solutions for differential equations with reflection of the argument, Nonlinear Anal.TAM, **57**(4)(2004) 633-637.
7. D.Piao, Pseudo almost periodic solutions for differential equations involving reflection of the argument, J. Korean Math. Soc. **41**(2004), 747-754.
8. A.M.Fink, Almost periodic differential equations, Lecture Notes in Math.377, Springer-Verlag, 1974.
9. C.Corduneanu, Almost periodic functions, Second Edition, Chelsea Publ. Comp., New York, 1989.
10. B.M.Levitan, V.V.Zhikov, Almost periodic differential functions and differential equations, Combridge: Combridge University Press, 1982.
11. W.A.Coppel, Almost periodic properties of ordinary differential equations, Ann.Math.Pura.Appl, **76** (1976), 27-50.
12. A.Cabada, F. Adrian F. Tojo, Comparison results for first order linear operators with reflection and periodic boundary value conditions, Nonlinear Anal. TMA,78(2013),32-46.

[†]SCHOOL OF MATHEMATICAL SCIENCES, OCEAN UNIVERSITY OF CHINA, QINGDAO, 266071, CHINA

[‡]DEPARTMENT OF MATHEMATICS, NORMAL COLLEGE, QINGDAO UNIVERSITY, QINGDAO, 266071, CHINA

E-mail address: dxpiao@ouc.edu.cn (D.Piao) xinna1026@sina.com(Na Xin)